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HECKE ACTIONS ON PICARD GROUPS

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Recently Robert Perlis [9] has found an intriguing application of the representation theory of finite permutation groups to number theory: Let K be a finite Galois extension of $\mathbb{Q}$ with Galois group G; let H, H' be subgroups of G and F, F' their corresponding fixed fields. Perlis is able to associate with every $\mathbb{Z}G$-homomorphism $\phi : \mathbb{Z}G/H \to \mathbb{Z}G/H'$ (of the permutation modules obtained from the action of G on the coset spaces G/H, G/H') a corresponding homomorphism from the class group of F' to that of F. This procedure is sufficiently powerful to prove, for example, that if ϕ becomes an isomorphism upon localization at a prime p, then the class groups of F and F' have isomorphic Sylow p-subgroups.

Perlis' method is fairly involved. First he considers the standard basis elements for $\mathrm{Hom}_G(\mathbb{Z}G/H, \mathbb{Z}G/H')$, corresponding to double cosets HgH' in G. For each such basis element he builds a map on class groups using conjugation and norm operators. (More accurately, he first redoes part of classical norm theory, relying on the capability of picking two special generators for ideals in number fields. However, this part of his argument could be replaced by a reference to [12] or [13].) Next Perlis separates each member of $\mathrm{Hom}_G(\mathbb{Z}G/H, \mathbb{Z}G/H')$ into a 'positive' and 'negative' part, using coefficients of standard basis elements. He constructs a map on class groups for each part individually, just by using products and the previously constructed maps on basis elements, and finally considers both parts together.

Our feeling was that there must be some more transparent explanation for Perlis' construction, perhaps related to the fact that the ring $\mathrm{Hom}_G(\mathbb{Z}G/H, \mathbb{Z}G/H)$ acts on $M^H \cong \mathrm{Hom}_G(\mathbb{Z}G/H, M)$ for any $\mathbb{Z}G$-module M. We have indeed found such an explanation, and the more formal proofs which are involved allow Perlis' method to be extended to arbitrary commutative rings (Theorem 3.4) and in most cases to schemes (Corollaries 3.5, 3.6). Moreover, we make a very simple observation (Remark 2.3) which considerably broadens the usefulness of Perlis' method even in the number field case. Unknown to Perlis, Nehrkorn [18] and Walter [19] had used

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similar but somewhat cruder methods to prove reduction theorems for the computation of class groups. We show how to obtain their results from the Perlis method and go beyond them. For example, we show that any Sylow p-subgroup of the class group of an abelian extension $K/\mathbb{Q}$ is computable from the class groups of subextensions with Galois group a direct product of a cyclic group with a group of order a power of p. Nehrkorn had proved the same result for the case $p \nmid [K:\mathbb{Q}]$. (We mention for the reader primarily interested in abelian extensions that there is a shortcut approach to the Perlis method in this case, described in the paragraph preceding Corollary 2.2.)

Our generalization beyond commutative rings to schemes is quite essential to our point of view, for it is only in the geometric analogues of the class group that the arguments become truly transparent. This paper has been organized so that the first two sections very quickly make it conceptually clear what is going on, while requisite technicalities from commutative algebra have been collected in Section 3 (along with the final form of some of the main results). Additional remarks have been collected in Section 4, and we invite the casual reader to browse through them.

1. The Hecke category

Let G be any abstract group. If H is a subgroup of G we let $\mathbb{Z}G/H$ denote the permutation module obtained from the action of G on the cosets gH with $g \in G$. Equivalently, $\mathbb{Z}G/H \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}$. Let $\mathscr{H}_G$ denote the category whose objects are the permutation modules $\mathbb{Z}G/H$ with H a subgroup of G, and whose morphisms are $\mathbb{Z}G$-module homomorphisms. In the spirit of Yoshida [14], we call $\mathscr{H}_G$ the Hecke category associated with G (though Yoshida reserves this name for a special subcategory when G is infinite). The following property of $\mathscr{H}_G$ is basic.

1.1. Proposition. [14]. *Let M be a $\mathbb{Z}G$-module. Then there is a contravariant additive function*

$$\Phi_M : \mathscr{H}_G \to \text{abelian groups}$$

with $\Phi_M(\mathbb{Z}G/H) = M^H$, the fixed points of H in M, for each subgroup H of G.

Proof. We have $M^H \cong \mathrm{Hom}_H(\mathbb{Z}, M) \cong \mathrm{Hom}_G(\mathbb{Z}G \oplus_{\mathbb{Z}H} \mathbb{Z}, M)$, and the result follows.

An important additional comment is that $(\mathbb{Z}G/H, M) \to M^H$ defines a bifunctor, an additive functor of two variables. Thus the assignment of $\mathbb{Z}G/H$ to $(\ )^H$ (and of M to Φ_M) is a functor to functors.

This gives in particular the following result on sheaves.

1.2. Lemma. *Let $\mathscr{M}$ be a sheaf of $\mathbb{Z}G$-modules on a topological space S. For H a subgroup of G let $\mathscr{M}^H$ denote the sheaf with $\mathscr{M}^H(U) = \mathscr{M}(U)^H$ for U open in S. Then* *there is a contravariant additive functor*

$$\Phi_{\mathcal{M}} : \mathscr{H}_G \to \textit{abelian sheaves}$$

with $\Phi_{\mathcal{M}}(\mathbb{Z}G/H) = \mathcal{M}^H$.

The same result holds for presheaves, *mutatis mutandis*. In the sequel we will relax notation and just write Φ for Φ_M, $\Phi_{\mathcal{M}}$ and similarly defined functors.

2. Some relative Picard groups

Let $S = (S, \mathcal{O}_S)$ be a ringed space and let $\mathscr{A}$ be a commutative $\mathcal{O}_S$-algebra. That is, S is a topological space equipped with a sheaf $\mathcal{O}_S$ of commutative rings, and $\mathscr{A}$ is a sheaf of commutative rings over S equipped with a map $\mathcal{O}_S \to \mathscr{A}$. The notion of an $\mathscr{A}$-module $\mathcal{M}$ is the obvious one. Let $\mathcal{M}|_U$ denote the sheaf induced by $\mathcal{M}$ on an open set U, and define S-Pic $\mathscr{A}$ to be the collection of isomorphism classes $[\mathcal{M}]$ of $\mathscr{A}$-modules $\mathcal{M}$ with $\mathcal{M}|_U \cong \mathscr{A}|_U$ as $\mathscr{A}|_U$-modules for all U is some open cover of S. These classes $[\mathcal{M}]$ form an abelian group under tensor product, and S-Pic $\mathcal{O}_S$ is the usual group Pic S. The group S-Pic $\mathscr{A}$ is similar to Fröhlich's 'locally free' Picard group for orders [15, §5], in that localization is considered only on the base. let $\mathscr{A}^*$ denote the sheaf of units of $\mathscr{A}$. Then the standard argument [7, III, Exercise 4.5] which shows Pic $S \cong H^1(S, \mathcal{O}_S^*)$ shows as well that S-Pic $\mathscr{A} \cong H^1(S, \mathscr{A}^*)$.

2.1. Proposition. *Let* $S = (S, \mathcal{O}_S)$ *be a ringed space,* $\mathscr{A}$ *a commutative* $\mathcal{O}_S$*-algebra, and* G *a group acting as* $\mathcal{O}_S$*-algebra automorphisms of* $\mathscr{A}$*. Then there is a contravariant additive functor*

$$\Phi : \mathscr{H}_G \to \textit{abelian groups}$$

with $\Phi(\mathbb{Z}G/H) = S\text{-Pic}\,\mathscr{A}^H$ *for each subgroup* H *of* G.

Proof. Since $(\mathscr{A}^H)^* = (\mathscr{A}^*)^H$, the result follows by composing $H^1(S, -)$ with the functor of Lemma 1.2 arising from the action of G on $\mathscr{A}^*$.

For the Perlis situation we take $S = \operatorname{Spec} \mathbb{Z}$ and $\mathscr{A}$ obtained from the ring of integers in a finite Galois extension $K/\mathbb{Q}$ by localization at the open sets of S. The group G is the Galois group of $K/\mathbb{Q}$. It is not hard to see that S-Pic $\mathscr{A}^H$ identifies with the class group of K^H; in any event this follows from Section 3. This gives the following corollary, which is implicit in Perlis [9] and summarizes his method. For a very quick proof for abelian extensions, let J be the idèle group of K and U the subgroup of elements which are units at finite primes; then just note that the class group of K^H is the cokernel of $(U \times K^*)^H \to J^H$ and apply (1.1). This doesn't work for nonabelian extensions because the multiplication map $U \times K^* \to J$ is not equivariant. (This difficulty could be remedied by appealing to similar but more sophisticated formulas for class groups of orders, cf. [17, 2.1], which incorporate the essential 'relative' localization feature of S-Pic $\mathscr{A}$.)

2.2. Corollary. *Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group G. Then there is a contravariant additive functor*

$$\Phi : \mathscr{H}_G \to \textit{abelian groups}$$

with $\Phi(\mathbb{Z}G/H) =$ class group of K^H for each subgroup H of G.

Observe that if $\mathscr{C}$ is any category with $\mathrm{Hom}_{\mathscr{C}}(M,N)$ an abelian group for each M,N in $\mathscr{C}$, and if F is any additive functor from $\mathscr{C}$ to the category of abelian groups, there is a formal way to 'tensor' both $\mathscr{C}$ and F with any commutative ring k: Define $k\otimes\mathscr{C}$ to be the category whose objects correspond bijectively to those in $\mathscr{C}$, and are formally labeled $k\otimes M$ with M ranging over $\mathscr{C}$. Define morphisms in $k\otimes\mathscr{C}$ by

$$\mathrm{Hom}_{k\otimes\mathscr{C}}(k\otimes M, k\otimes N) = k\otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(M,N).$$

Thus $k\otimes\mathscr{C}$ is a category with $\mathrm{Hom}_{k\otimes\mathscr{C}}(k\otimes M, k\otimes N)$ a k-module for each pair of objects $k\otimes M$, $k\otimes N$ in $k\otimes\mathscr{C}$. We have a k-linear functor

$$k\otimes F : k\otimes\mathscr{C} \to k\text{-modules}$$

with $(k\otimes F)(k\otimes M) = k\otimes_{\mathbb{Z}} F(M)$, and with $(k\otimes F)(\alpha\otimes_{\mathbb{Z}} f) = \alpha\otimes_{\mathbb{Z}} F(f)$ for any $\alpha\in k$ and $f\in \mathrm{Hom}_{\mathscr{C}}(M,N)$.

Thus we always get a functor to abelian groups from the formally defined category $k\otimes\mathscr{C}$, and the issue now is identification of the latter in favorable circumstances. *When $\mathscr{C}$ is a category $\mathscr{H}_G$ as in Section 1, then $k\otimes\mathscr{C}$ identifies naturally with the category of kG-modules of the form $k\otimes_{\mathbb{Z}} M$ for M in $\mathscr{C}$.* We defer a proof of this for arbitrary G and k to 4.2. For now the reader can see this is true for G finite and k flat over $\mathbb{Z}$ (additively torsion free) from well-known properties of finitely presented modules.

The Perlis result regarding Sylow p-subgroups mentioned in the introduction is now a formal consequence of Corollary 2.2: Let p be a prime and let $\mathbb{Z}_p$ denote the complete ring of p-adic integers. From the functor Φ in 2.2 we can construct a functor $\mathbb{Z}_p\otimes\Phi$ as above on $\mathbb{Z}_p\otimes\mathscr{H}_G$. Thus $\mathbb{Z}_pG/H \approx \mathbb{Z}_pG/H'$ implies $\mathbb{Z}_p\otimes_{\mathbb{Z}}\Phi(\mathbb{Z}G/H) \approx \mathbb{Z}_p\otimes_{\mathbb{Z}}\Phi(\mathbb{Z}G/H')$. Of course $\mathbb{Z}_p\otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A when A is a finite abelian group; hence the class groups of K^H and $K^{H'}$ have isomorphic Sylow p-subgroups.

2.3. Remark. The utility of this method is enhanced considerably if the category $\mathscr{H}_G$ is replaced by the category $\dot{\mathscr{H}}_G$ of all finite direct sums of objects in $\mathscr{H}_G$. Any additive functor from $\mathscr{H}_G$ to an additive category $\mathscr{A}$ extends automatically to a functor on $\dot{\mathscr{H}}_G$, and there are many more interesting p-local isomorphisms between objects in $\dot{\mathscr{H}}_G$ then $\mathscr{H}_G$.

For example, suppose the group G contains an abelian subgroup H of order q^2 and exponent q for some prime $q \neq p$. Let $H_1, \ldots, H_{q+1}$ denote the subgroups of H of order q. Then we have an isomorphism of $\mathbb{C}H$-modules

$$\mathbb{C}H \oplus \mathbb{C}^{(q)} \approx \mathbb{C}H/H_1 \oplus \cdots \oplus \mathbb{C}H/H_{q+1}$$

which is easily checked by decomposing both sides into linear factors. The notation $\mathbb{C}^{(q)}$ means a direct sum of q copies of the trivial module $\mathbb{C}$. Since $p \nmid |H|$, standard results in representation theory allow us to replace $\mathbb{C}$ in the displayed isomorphism with $\mathbb{Z}_p$. Now, tensoring over $\mathbb{Z}_pH$ both sides with $\mathbb{Z}_pG$, we obtain the following isomorphism in $\mathbb{Z}_p \otimes \mathscr{H}_G$:

$$\mathbb{Z}_pG/1 \oplus (\mathbb{Z}_pG/H)^{(q)} \approx \mathbb{Z}_pG/H_1 \oplus \cdots \oplus \mathbb{Z}_pG/H_{q+1}.$$

Consequently if $G = \mathrm{Gal}(K/\mathbb{Q})$ we have

$$(\mathrm{Cl}\, K)_p \oplus (\mathrm{Cl}\, K^H)_p^{(q)} \approx (\mathrm{Cl}\, K^{H_1})_p \oplus \cdots \oplus (\mathrm{Cl}\, K^{H_{q+1}})_p$$

where the notation $(\mathrm{Cl}\, K)_p$ refers to the Sylow p-subgroup of the class group of K. If G is abelian, we can always find such a subgroup H unless G is the direct product of a p-group with a cyclic group.

2.4. Corollary. *Let $K/\mathbb{Q}$ be a finite abelian extension with Galois group G, and let p be a fixed prime. Then the Sylow p-subgroup of the class group of K may be computed in terms of G and Sylow p-subgroups of class groups for subfields of K with Galois group the direct product of a p-group with a cyclic group.*

As mentioned in the introduction, this result is due to Nehrkorn [18] in case $p \nmid |G|$. Our next corollary was first stated by Nehrkorn [18] and proved by Walter [19], but in both instances again with the assumption $p \nmid |G|$. (In which case $\mathbb{Z}_p$ may be replaced by $\mathbb{C}$ in the hypothesis.)

2.5. Corollary. *Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group G, and let p be a fixed prime. Let $H_1, \ldots, H_n$ be subgroups of G. Then any isomorphism*

$$\bigoplus_{i=1}^{n} (\mathbb{Z}_pG/H_i)^{(a_i)} \approx \bigoplus_{i=1}^{n} (\mathbb{Z}_pG/H_i)^{(b_i)}$$

with nonnegative integers a_i, b_i gives an isomorphism

$$\bigoplus_{i=1}^{n} (\mathrm{Cl}\, K^{H_i})_p^{(a_i)} \approx \bigoplus_{i=1}^{n} (\mathrm{Cl}\, K^{H_i})_p^{(b_i)}.$$

This is just an explicit statement of the main content of Remark 2.3, which we have already applied in the proof of Corollary 2.4. Here's another application based on the same principle and the induction theory of permutation modules developed by Conlon and Dress.

2.6. Corollary. *Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group G, and let p be a fixed prime. Then the Sylow p-subgroups of class groups of arbitrary subfields K^H with H a subgroup of G are computable from knowledge of G and the structure of these Sylow p-subgroups for all cases in which H is a cyclic extension of a p-subgroup.*

This follows from Corollary 2.5 and the fact [4, Prop. 9.2] that, for some integer $n>0$ and $\mathbb{Z}_pG$-modules M, N which are finite direct sums of modules of the form $\mathbb{Z}_pG/H$ with H a cyclic extension of a p-group, we have an isomorphism

$$\mathbb{Z}_p^{(n)} \oplus M \approx N$$

of $\mathbb{Z}_pG$-modules. (Tensoring with an arbitrary $\mathbb{Z}_pG/H$ gives the relation required in the hypothesis of Corollary 2.5. Alternately, one can just quote [4] for each $\mathbb{Z}_pG/H$.)

It is not always necessary to apply Corollary 2.2, Remark 2.3 for isomorphisms. A split surjection, for example, can yield interesting information. For an illustration, see Proposition 4.7.3 or 4.7.4.

Finally we mention that all the corollaries above have analogues in the theory of Picard groups of commutative rings once Theorem 3.4 is established, and similar remarks apply for schemes in most cases. For infinite Picard groups the reader may want to consider coefficients other than $\mathbb{Z}_p$, and should consult 4.2 for a discussion.

3. Absolute Picard groups

We require two further results on S-Pic $\mathscr{A}$. For terminology regarding schemes we refer the reader to Hartshorne [7].

3.1. Lemma. *Let $S=(S, \mathcal{O}_S)$ be a scheme and $\mathscr{A}$ a quasicoherent $\mathcal{O}_S$-algebra. Then the following are equivalent for an $\mathscr{A}$-module $\mathscr{M}$ over S:*

(a) $[\mathscr{M}] \in S$-Pic $\mathscr{A}$.

(b) *$\mathscr{M}$ is quasicoherent as an $\mathcal{O}_S$-module, $\mathscr{M}_s \approx \mathscr{A}_s$ as $\mathscr{A}_s$-modules for each $s \in S$, and $\mathscr{M}(U)$ is a finitely generated $\mathscr{A}(U)$-module for each U in some open affine cover of S.*

Proof. Clearly (a)$\Rightarrow$(b). ($\mathscr{M}(U)$ is finitely generated on *any* open affinte U, since any open cover of an affine scheme contains a finite subcover.) To prove the converse we can assume $S=\operatorname{Spec} k$ for a commutative ring k and that $\mathscr{A}, \mathscr{M}$ are obtained by localization from a commutative k-algebra A and one of its finitely generated modules M. Fix $s \in S$ and let $\psi: A_s \to M_s$ be an isomorphism. Adjusting denominators we can assume $\psi(1)=m_s$, the image in M_s of some $m \in M$. Let $\phi: A \to M$ be the map $a \mapsto am$, and let ϕ_t denote its localization at $t \in S$; thus $\phi_s=\psi$. Let $N \subseteq M$ be a finite set of generators for M as an A-module. Then for given $t \in S$, the map

$\phi_t : A_t \to M_t$ is surjective if and only if $N_t \subseteq (\phi(F)m)_t$ for some finitely generated k-submodule F of A, or equivalently $(kn/kn \cap \phi(F)m)_t = 0$ for each $n \in N$. For given F, the set of points $t \in S$ satisfying the condition is clearly open in $S = \operatorname{Spec} k$. Hence the set of points $t \in S$ for which ϕ_t is surjective form an open set, hence contains an open affine subset U containing s. The surjective morphism $\phi_U : \mathcal{M}(U) \to \mathcal{A}(U)$ of $\mathcal{A}(U)$-modules of course splits, and is an isomorphism at s. Thus ϕ_U is an isomorphism, and the lemma is proved.

The lemma above of course parallels the well-known 'local characterization of finitely generated projective modules' [2, II, §5, Theorem 1]. We also note that the lemma could be phrased completely geometrically, since each quasicoherent $\mathcal{O}_S$-algebra $\mathcal{A}$ gives rise to a scheme $X = \operatorname{Spec} \mathcal{A}$ affine over S, and conversely, if $f : X \to S$ is an affine morphism of schemes, then the direct image $f_* \mathcal{O}_X$ is a quasicoherent $\mathcal{O}_S$-algebra [7, II, Exercise 5.17].

3.2. Lemma. *Let $f : X \to S$ be an affine morphism of schemes with $(f_* \mathcal{O}_X)_s$ semilocal for each s in S. Let $\mathcal{N}$ be an $\mathcal{O}_X$-module. Then $[f_* \mathcal{N}] \in S\text{-}\operatorname{Pic} f_* \mathcal{O}_X$ iff $[\mathcal{N}] \in \operatorname{Pic} X$, and the resulting correspondence is an isomorphism $S\text{-}\operatorname{Pic} f_* \mathcal{O}_X \cong \operatorname{Pic} X$.*

Proof. The isomorphism will follow from the first assertion, since $\mathcal{N} \mapsto f_* \mathcal{N}$ gives an equivalence on the quasicoherent module categories for $\mathcal{O}_X$ and $f_* \mathcal{O}_X$ [7, *op. cit.*]. Of course $[f_* \mathcal{N}] \in S\text{-}\operatorname{Pic} f_* \mathcal{O}_X$ implies $[\mathcal{N}] \in \operatorname{Pic} X$. To prove the converse, we can localize and assume both S and X are affine; consequently $\mathcal{N}(X)$ and thus $(f_* \mathcal{N})_s = \mathcal{N}(X)_s$ are finitely generated $\mathcal{O}(X)$ and $(f_* \mathcal{O}_X)_s = \mathcal{O}(X)_s$ modules, respectively. Moreover, $\mathcal{N}(X)$ and $(f_* \mathcal{N})_s$ are projective [2, *op. cit.*] and rank 1. Since $(f_* \mathcal{O}_X)_s$ is semilocal, $(f_* \mathcal{N})_s$ is free [2, II, §5, Prop. 5] and thus isomorphic to $(f_* \mathcal{O}_X)_s$. By Lemma 3.1 we have $f_* \mathcal{N} \in S\text{-}\operatorname{Pic} f_* \mathcal{O}_X$.

Perhaps more to the point is to give conditions under which the hypothesis of Lemma 3.2 are satisfied:

3.3. Lemma. *Let $f : X \to S$ be a morphism of schemes. Assume either*

(a) *f is of finite type, or*

(b) *There is a scheme T, a quasicoherent $\mathcal{O}_T$-algebra $\mathcal{A}$, a group G acting as $\mathcal{O}_T$-algebra automorphisms of $\mathcal{A}$, and a subgroup H of finite index in G such that*

(i) *$\mathcal{A}^H, \mathcal{A}^G$ are both quasicoherent*[1]*;*

(ii) *$X = \operatorname{Spec} \mathcal{A}^H, S = \operatorname{Spec} \mathcal{A}^G$; and*

(iii) *f is the map obtained from the inclusion $\mathcal{A}^G \subseteq \mathcal{A}^H$.*

Then f is affine, and $(f_* \mathcal{O}_X)_s$ is semilocal for each $s \in S$ (that is, the hypothesis of Lemma 3.2 *is satisfied).*

[1] This means, for $\mathcal{A}^G$ say, that $(\mathcal{A}(U)_v)^G = (\mathcal{A}(U)^G)_v$ for U open affine in T and $v \in \mathcal{O}(U)$. This condition is almost always satisfied (cf. Corollary 3.5, 4.5).

Proof. In case (a) the map f is affine by definition, and every maximal ideal of $(f_*\mathcal{O}_X)_s$ contains the image of the maximal ideal of $(\mathcal{O}_S)_s$ by the going-up theorem. Hence these ideals correspond bijectively to the maximal ideals in a finite-dimensional algebra, and so $(f_*\mathcal{O}_X)_s$ is semilocal.

In case (b) let U be open affine in T, and set $A=\mathcal{A}(U)$, $B=\mathcal{A}(U)^H$, and $k=\mathcal{A}(U)^G$. By assumption $V=\operatorname{Spec} k$ is open in S, and $f^{-1}(V)=\operatorname{Spec} B$ is affine. Since S is covered by such open sets V, the morphism f is affine. To prove $(f_*\mathcal{O}_X)_s$ is semilocal, we can assume $S=\operatorname{Spec} k$.

Let K be the intersection of all G conjugates of H. Without loss we can replace A above by A^K, and H, G by H/K, G/K, respectively. In other words, we can assume G is finite.

Note that A, and hence B, is integral over k by a classical argument: each $a\in A$ satisfies $p_a(a)=0$, where $p_a(t)=\prod_{g\in G}(t-{}^g a)$ is monic with coefficients in $A^G=k$. Thus all maximal ideals of B_s contain the maximal ideal of k_s by the going-up theorem, and it suffices now just to prove $f^{-1}(s)$ is finite.

Let e be the map $\operatorname{Spec} A\to\operatorname{Spec} B$, and let d be the composition fe. Invoking integrality again, we obtain that e is surjective. Thus it is enought to show $d^{-1}(s)$ is finite.

In fact it is well-known that more is true – this fiber is a single orbit of G: Choose $x\in d^{-1}(s)$ closed in the relative topology, and suppose $y\in d^{-1}(s)$ does not belong to the orbit of x. Then there is an open set W in $\operatorname{Spec} A$ containing y which has empty intersection with the orbit of x. Without loss $W=\operatorname{Spec} A_w$ for $w\in A$. In terms of prime ideals we have $w\notin\mathfrak{p}_y$ and ${}^g w\in\mathfrak{p}_x$ for each $g\in G$. The latter condition implies that all coefficients of the polynomial $p_w(t)$ defined above except for the coefficient of the highest power of t, belong to $\mathfrak{p}_x\cap k=\mathfrak{p}_s$. Since $p_w(w)=0$, some power of w belongs to $\mathfrak{p}_sA\subseteq\mathfrak{p}_y$, and so $w\in\mathfrak{p}_y$, a contradiction. This completes the proof of the lemma.

If A is a commutative ring, we write $\operatorname{Pic} A$ for $\operatorname{Pic}(\operatorname{Spec} A)$; this is the traditional group of isomorphism classes of invertible A-modules, rank 1 projective A-modules, etc. [1].

3.4. Theorem. *Let G be a group of automorphisms of a commutative ring A. Then there is a contravariant additive functor*

$$\Phi:\mathcal{H}_G\to \textit{abelian groups}$$

with $\Phi(\mathbb{Z}G/H)=\operatorname{Pic} A^H$ for each subgroup H of G.'

Proof. We give full details for G finite, and just sketch the rest. By Proposition 2.1 we have a contravariant additive functor Φ with $\Phi(\mathbb{Z}G/H)=S\text{-}\operatorname{Pic}\mathcal{A}^H$ where $S=\operatorname{Spec} A^G$ and $\mathcal{A}$ is the direct image on S of the structure sheaf of $\operatorname{Spec} A$. We claim $\mathcal{A}^H$ is quasicoherent – that is, $(A_f)^H=(A^H)_f$ for each $f\in A$. This will establish the hypothesis of Lemma 3.3 with $S=T$, and we can conclude from Lemma 3.2 that

S-Pic $\mathscr{A}^H \cong \operatorname{Pic} A^H$. Clearly $(A^H)_f \le (A_f)^H$, and if $a/f^n \in A_f$ is fixed by H, then, for each $h \in H$, the element $a - {}^h a$ is annihilated by some power of f. Since H is finite some power f^m of f serves for all h, so $a/f^n = af^m/f^{n+m} \in (A^H)_f$. This completes the proof for the finite case.

In the infinite case we need to use 'generators and relations' for the category $\mathscr{H}_G^{\mathrm{op}}$ dual to $\mathscr{H}_G$. For H,K subgroups of G, there is an obvious basis for $\operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G/H, \mathbb{Z}G/K)$ corresponding to double cosets HxK with $[H : H \cap xKx^{-1}] < \infty$. The morphism corresponding to HxK may be further decomposed in $\mathscr{H}_G^{\mathrm{op}}$ as a composition of 'conjugation' by x, a 'restriction' from xKx^{-1} to $xKx^{-1} \cap H$, and 'induction' (or 'norm') from $H \cap xKx^{-1}$ to H. (Our discussion here is a variation on a theme of Yoshida [14], where the reader may find some further details.) Among the relations which must be satisfied, the most important is the 'Mackey formula': if $K \subseteq H$ with $[H:K] < \infty$ and L is any subgroup of H, then for $\gamma \in \mathbb{Z}G/K$ we have, in an obvious notation

$$\gamma|^H|_L = \sum_{LxK} ({}^x\gamma)|_{xHx^{-1} \cap L}|^L.$$

To get the desired functor Φ one has to give values of Φ on conjugation, restriction, and induction morphisms and check that the appropriate relations are satisfied. Conjugation and restriction are easy, and induction can be obtained from the finite group case of the theorem (first replacing A by A^N where N is a normal subgroup of H with $K \supseteq N$ and $[H:N] < \infty$). To check, for example, the above Mackey formula, represent an element in $\operatorname{Pic} A^K$ by a Čech 1-cocycle $\gamma = \{\gamma_{u,v}\}$ over $\operatorname{Spec} A^H$, using Lemma 3.2, with the $\gamma_{u,v}$ units in suitable localizations of A^K. If X is a set of representatives for the cosets xK in H, then $\{\prod_{x \in X} {}^x\gamma_{u,v}\}$ represents our Φ version of 'induction' applied to $[\gamma]$; the further 'restriction' to $\operatorname{Pic} A^L$ is obtained just by replacing all u, v with their inverse images u', v' under $\operatorname{Spec} A_L \to \operatorname{Spec} A^H$. Now the set X is a disjoint union of $(X \cap LxK)$'s, and we can choose X to begin with so that $(X \cap LxK)x^{-1}$ is a set Y_x of representatives for the cosets $y(xKx^{-1} \cap L)$ in L. Clearly the 1-cocycle on $\operatorname{Spec} A_L$ assigning u', v' to $\prod_{y \in Y_x} {}^{yx}\gamma_{u,v}$ represents in $\operatorname{Pic} A^L$ the class corresponding to the image of $[\gamma]$ under 'conjugation' by x, followed by 'restriction' to $\operatorname{Pic} A^{(xKx^{-1} \cap L)}$ and 'induction' to $\operatorname{Pic} A^L$. Since $\prod_{x \in X} {}^x\gamma_{u,v} = \prod_{LxK} \prod_{y \in Y_x} {}^{yx}\gamma_{u,v}$, the Mackey formula is verified. We leave further details to the reader.

As corollaries of the proof, we note the following two results for schemes.

3.5. Corollary. *Let $f : X \to S$ be an affine morphism of schemes, and G a group of automorphisms of X/S. Assume X is locally noetherian or reduced, or that G is finite. Then there is a contravariant additive functor*

$$\Phi : \mathscr{H}_G \to \textit{abelian groups}$$

with $\Phi(\mathbb{Z}G/H) = \operatorname{Pic}(\operatorname{Spec}(f_* \mathscr{O}_X)^H)$.

Proof. Once one knows $(f_*\mathcal{O}_X)^H$ is quasicoherent, the proof proceeds as in the affine case. Let $U \subseteq S$ be open affine, and $\upsilon \in \mathcal{O}(U)$. Set $A = \mathcal{O}(f^{-1}(U))$; we want $(A_\upsilon)^H = (A^H)_\upsilon$. As in the proof of Theorem 3.4 it is enough to know, for $a \in A$, that whenever each element of $\{a - {}^h a \mid h \in H\}$ is annihilated by some power of υ, then a single power of υ works for all the elements. This is clear if annihilators in A satisfy the ascending chain condition, or if H is finitely generated. Hence $(f_*\mathcal{O}_X)^H$ is quasicoherent if X is locally noetherian or G is finite. The same conclusion holds if X is reduced, since $\upsilon(a - {}^h a)$ is nilpotent when $(a - {}^h a)$ is annihilated by some power of υ. (This last fact was pointed out by D. Costa, who suggested this version of the proof.)

The extra assumption that X is locally noetherian or reduced may be dropped for arbitrary G if one is content to consider a suitable subcategory of $\mathscr{H}_G$ – e.g. all $\mathbb{Z}G/H$ with H finitely generated.

The next corollary describes a common Galois theory situation for schemes. If X is a scheme over a commutative ring k and A is a commutative k-algebra, we abbreviate $\operatorname{Spec} A \times_{\operatorname{Spec} k} X$ by $A \otimes_k X$.

3.6. Corollary. *Let k be an integral domain, A a domain integral over k, and G the group of all automorphisms of A fixing each element of k. Let X be any scheme flat over k. Then there is a contravariant additive functor*

$$\Phi : \mathscr{H}_G \to \textit{abelian groups}$$

with $\Phi(\mathbb{Z}G/H) = \operatorname{Pic} A^H \otimes_k X$ for each subgroup H of G.

Proof. Let $\mathscr{A}$ be the direct image over X of the structure sheaf of $A \otimes_k X$. We claim for any subgroup H of G that $\mathscr{A}^H$ is quasicoherent and $\operatorname{Spec} \mathscr{A}^H \cong A^H \otimes_k X$. The corollary will then follow from the proof of Theorem 3.4, as noted before.

Let U be open affine in X and $f \in \mathcal{O}(U)$. Put $B = A \otimes_k \mathcal{O}(U)$, and regard $B \subseteq K \otimes_k \mathcal{O}(U)$ where K is the quotient field of A. Clearly any $b \in B$ belongs to $F \otimes_k \mathcal{O}(U)$ for some field $F \subseteq K$ which is a finite extension of the quotient field of k. Thus b is fixed by a subgroup of finite index in G. It follows easily that a single power of f suffices to kill all elements of the form $b - {}^g b$, where $g \in G$, that are killed by any power of f at all. As before, this gives $(B_f)^H = (B^H)_f$, so $\mathscr{A}^H$ is quasicoherent.

Similarly one can argue using flatness that $b \in B^H$ iff $b \in A^H \otimes_k \mathcal{O}(U)$. This identification $\mathscr{A}^H(U) = A^H \otimes_k \mathcal{O}(U)$ gives the desired isomorphism $\operatorname{Spec} \mathscr{A}^H \cong A^H \otimes_k X$, and the proof is complete.

4. Concluding remarks

We have collected in this section a number of remarks not essential for the main line of exposition.

4.1. *Noncommutative theory.* If $\mathscr{A}$ is a sheaf of noncommutative rings over a ringed space $S=(S,\mathcal{O}_S)$, then one can still define S-Pic $\mathscr{A}$ using bimodules locally isomorphic to $\mathscr{A}$ on S. The result of this is that S-Pic $\mathscr{A}\cong S$-Pic $\mathscr{Z}$ where $\mathscr{Z}$ assigns to an open U in S the center $Z(\mathscr{A}(U))$ of $\mathscr{A}(U)$. The identification is accomplished through $H^1(S,\mathscr{Z}^*)$. Thus we are led back to sheaves of commutative rings. There are a number of other ways to define a Picard group for bimodules, and we refer the interested reader to [15], [16], [17].

More interesting perhaps is to consider one-sided $\mathscr{A}$-modules locally isomorphic to $\mathscr{A}$ on S. We hope to make a detailed study of this question in a later paper, but for now we can indicate what we think is going on. First, there is still an identification of these isomorphism classes of onesided modules with the elements of $H^1(S,\mathscr{A}^*)$, though the latter is just a set and the theory of Section 1 does not apply. To get started, one has to approximate $H^1(S,\mathscr{A}^*)$ with an abelian group, and the natural choice is the image of $H^1(S,\mathscr{A}^*)$ in $H^1(S,K_1(\mathscr{A}))$. We expect this generalizes the locally free class group Cl Λ, cf. [17], for orders in semisimple algebras. There are still difficulties to overcome, largely due to the fact that K_1 does not in general commute with the fixed point functor of a group action. Nevertheless we expect this does occur often enough so that, if G is a Galois group acting on Λ through the coefficient domain, then there is a functor $\mathbb{Z}G/H\to \mathrm{TCl}\,\Lambda^H$ on $\mathscr{H}_G$ for a large factor TCl of the class group.

4.2. *General coefficients.* Let G be a group and k a commutative ring. The category $\mathscr{H}_{kG}$ is defined by taking permutation modules $kG/H\cong k\otimes\mathbb{Z}G/H$ as objects, with kG-module homomorphisms as morphisms. Because of the canonical basis referred to in the proof of Theorem 3.4, which serves for any k, we have $\mathrm{Hom}_{kG}(kG/H,kG/H')\cong k\otimes \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G/H,\mathbb{Z}G/H')$ for all subgroups H,H' of G. Consequently any contravariant additive functor

$$\Phi:\mathscr{H}_G\to\text{abelian groups}$$

defines in an obvious way a contravariant k-linear functor

$$\Phi_k:\mathscr{H}_{kG}\to k\text{-modules}$$

with $\Phi_k(kG/H)=k\otimes\Phi(\mathbb{Z}G/H)$. Similarly one defines the category $\hat{\mathscr{H}}_{kG}$ of finite direct sums of objects in $\mathscr{H}_{kG}$, or, equivalently, objects $k\otimes A$ with an object in $\hat{\mathscr{H}}_G$, and observes that Φ_k defines a functor here as well. This has the following consequence, which slightly extends the discussion in Corollary 2.2, Remark 2.3.

4.2.1. Proposition. *Let G be a group, k a commutative ring, and suppose we are given a contravariant additive functor $\Phi:\mathscr{H}_G\to$ abelian groups. Extend Φ in the obvious way to the category $\hat{\mathscr{H}}_G$ of direct sums of objects in $\mathscr{H}_G$. Then for any objects A,B in $\hat{\mathscr{H}}_G$, if $k\otimes A\approx k\otimes B$, then $k\otimes\Phi(A)\approx k\otimes\Phi(B)$.*

We also note the following useful fact:

4.2.2. *Let G be a finite group and A, B objects in $\mathscr{H}_G$. Let p be a fixed prime and let $\mathbb{F}_p, \mathbb{Z}_p$, and $\mathbb{Z}_{(p)}$ denote the field of p elements, the complete p-adic integers, and the localization of the ring $\mathbb{Z}$ at the prime ideal (p), respectively. Then if $\mathbb{Z}_pA \approx \mathbb{Z}_pB$ or $\mathbb{F}_pA \approx \mathbb{F}_pB$, we have $\mathbb{Z}_{(p)}A \approx \mathbb{Z}_{(p)}B$.*

The proof is obtained easily from the isomorphism

$$\mathbb{F}_p \otimes \mathrm{Hom}_{\mathbb{Z}G}(A, B) \cong \mathrm{Hom}_{\mathbb{F}_pG}(\mathbb{F}_p \otimes A, \mathbb{F}_p \otimes B)$$

and the finite $\mathbb{Z}$-rank of A and B.

Finally we mention that both Proposition 4.2.1 and 4.2.2 remain true if "$\approx$" is replaced everywhere by "is a direct sumand of". The proofs are essentially the same.

4.3. *Classical norm theory.* Let G be a group and $\Phi : \mathscr{H}_G \to$ abelian groups a contravariant additive functor. Let $H \subseteq K$ be subgroups of G with $[K:H] < \infty$. The diagonal map $\mathbb{Z} \to \mathbb{Z}K/H$ produces a map $\eta_{H,K} : \mathbb{Z}G/K \to \mathbb{Z}G/H$ and consequently a map $N_{K,H} = \phi(\eta_{H,K}) : \Phi(\mathbb{Z}G/H) \to \Phi(\mathbb{Z}G/K)$. In the Perlis situation this is the classical norm map on ideal classes [12, 4–7–6], [13, p. 53]. In all the situations we have discussed where a functor Φ is available the map $N_{K,H}$ gives a general version of this classical norm theory, the usual formal properties being easily read off from the functorial properties of Φ. For a list, see Yoshida [14]: one defines also 'conjugation' and 'restriction' maps (as we used for the infinite case in Theorem 3.4), and the properties satisfied by the three operations are summarized with Green's notion of a cohomological G-functor [5].

We take this opportunity to point that an alternate treatment of the Perlis' results could be obtained just by checking the axioms for a cohomological G-functor with the classical norm theory cited above. In some sense the norm is the only 'fundamental' ingredient in Perlis' results, though the norm concept hardly suggests the kind of application discussed in Section 2.

4.4. *The positive cone of $\mathscr{H}_G$.* This paragraph formalizes an interesting technique of Perlis not needed elsewhere in this paper. By taking only positive linear combinations of members of the basis referred to in Theorem 3.4 and 4.2, we obtain a subcategory $\mathscr{H}_G^+$ with $\mathrm{Hom}_{\mathscr{H}_G^+}(A, B)$ a subsemigroup of $\mathrm{Hom}_{\mathscr{H}_G}(A, B)$, for A, B objects in $\mathscr{H}_G$. Moreover the second Hom is the groupification of the first. This implies that any additive functor from $\mathscr{H}_G^+$ to an additive category extends uniquely to an additive functor on H. Additive functors from $\mathscr{H}_G^+$ to the category of abelian semigroups may be obtained by taking fixed points on an abelian semigroup with G-action analogously to Proposition 1.1, and functors to an additive category may subsequently arise from these. The main application we have in mind is given below. If A is a commutative algebra over a commutative ring k, we write k-Pic A for the group S-Pic $\mathscr{A}$ where $S = \mathrm{Spec}\, k$ and $\mathscr{A}$ is obtained from A by localization.

4.4.1. Proposition. *Let A be a commutative algebra over a commutative ring k, and let G be a group of k-algebra automorphisms of A. Then there is a contravariant additive functor*

$$\Phi : \mathscr{H}_G \to \textit{abelian groups}$$

with $\Phi(\mathbb{Z}G/H) = k\text{-Pic}\, A^H$ for each subgroup H of G.

For the proof, let $\mathscr{A}$ be the sheaf of commutative rings over Spec k obtained from A by localization. The sheaf which defines k-Pic A^H is obtained by taking the image of the constant presheaf A^H in $\mathscr{A}^H$, then taking the multiplicative groups generated over each open set U by elements invertible in $\mathscr{A}^H(U)^*$, and finally sheafifying the resulting presheaf of abelian groups. One gets easily by arguing with abelian semigroups that the assignment of $\mathbb{Z}G/H$ to this sheaf defines a contravariant additive functor on $\mathscr{H}_G^+$. But, since we ended up with sheaves of abelian groups, we get an additive functor on $\mathscr{H}_G$ as well, and the proposition follows.

4.5. *An example.* Let A be a commutative algebra over a commutative ring k, and H a group of k-algebra automorphisms of A. If $f \in k$ then it is *not* true in general that $(A^H)_f = (A_f)^H$, in spite of the fact that this does indeed hold in most common situations. (E.g., if A is noetherian or reduced, or if H is finitely generated; see the proof of Corollary 3.5. We mention also that the analogous statement for action of affine group schemes over k is always true.) For an example, let Q be any field, let k be the polynomial ring $Q[f]$ in one indeterminant f, and let A be the commutative k-algebra generated by elements $x, y_1, y_2, \ldots$ subject to relations $y_i y_j = 0$ for all i, j and $fy_i = y_{i-1}$ (define $y_0 = 0$). For a positive integer n, let h_n be the k-algebra automorphism with $h_n(x) = x + y_n$ and $h_n(y_i) = y_i$ for all i, and let H be the group generated by $h_1, h_2, \ldots$. Verification that h_n is well defined is of course trivial to do at the Q-algebra level.

Let $\bar{x}$ denote the image of x in A_f, and note that all of $A_f = k_f[\bar{x}]$ is fixed by H. (Each y_i is annihilated by a power of f.) On the other hand $\bar{x} \notin (A^H)_f$. Otherwise, using the fact that all powers of f are fixed by H, we get an element $f^n x \in A^H$ for some positive integer n. However $h_m(f^n x) \neq f^n x$ for $m > n$, so this is a contradiction. Thus $(A^H)_f \neq (A_f)^H$.

We note that the group H in this example is free abelian on the generators $h_1, h_2, \ldots$.

4.6. *Hecke actions.* The title of this paper is derived from the fact that if G is a group and

$$\Phi : \mathscr{H}_G \to \text{abelian groups}$$

is a contravariant additive functor, then the endomorphism ring, or 'Hecke algebra', associated with $\mathbb{Z}G/H$, for H a subgroup of G, *acts as a ring* on $\Phi(\mathbb{Z}G/H)$. The functor Φ itself may be viewed as just defining a module $\bigoplus_H \Phi(\mathbb{Z}G/H)$ for a big ring $\bigoplus_{H,H'} \mathrm{Hom}_G(\mathbb{Z}G/H, \mathbb{Z}G/H')$ (attach an identity in the usual way if G is infinite), though this is clumsy, and the applications in Section 2 in any event indicate the point of view of functors.

It is interesting to note that the category of such functors extends the category of G-modules via Proposition 1.1. This seems to offer a rich environment for derived functor theories, though we have not investigated these possibilities in any detail.

For the recent history of Hecke algebra actions, see Yoshida [14].

4.7. *The map* $\mathrm{Pic}\, A^G \to \mathrm{Pic}\, A$. This paragraph is motivated by results of Samuel and Kang, cf. [8]. Also, we would like to thank David Carter for some helpful remarks and for showing us some of his unpublished work.

4.7.1. Proposition. *Let $S=(S,\mathcal{O}_S)$ be a ringed space, $\mathcal{A}$ an $\mathcal{O}_S$-algebra, and G a group acting as $\mathcal{O}_S$-algebra automorphisms of $\mathcal{A}$. Then the kernel of* S-$\mathrm{Pic}\,\mathcal{A}^G \to S$-$\mathrm{Pic}\,\mathcal{A}$ *is naturally isomorphic to the subgroup of $H^1(G,\mathcal{A}(S)^*)$ consisting of locally trivial classes, that is, the kernel of*

$$H^1(G,\mathcal{A}(S)^*) \to \prod_{x\in S} \varinjlim_{x\in V(\text{open})} H^1(G,\mathcal{A}(V)^*).$$

4.7.2. Corollary. *Let A be a commutative ring and G a group acting on A as a group of ring automorphisms. Then the kernel of* $\mathrm{Pic}\, A^G \to \mathrm{Pic}\, A$ *is naturally isomorphic to the subgroup of $H^1(G,A^*)$ consisting of locally trivially classes, that is, the kernel of*

$$H^1(G,A^*) \to \prod_{x\in \mathrm{Spec}\, A} \varinjlim_{f\in A, f\notin x} H^1(G,A_f^*).$$

The corollary is immediate from the proposition, taking $S=\mathrm{Spec}\, A^G$. (The map $\mathrm{Pic}\, A^G \to \mathrm{Pic}\, A$ factors as S-$\mathrm{Pic}\,\mathcal{A}^G \to S$-$\mathrm{Pic}\,\mathcal{A} \subseteq \mathrm{Pic}\,\mathcal{A}$, where $\mathcal{A}$ is obtained from A by localization.)

We sketch two proofs for the proposition itself. First, one can just represent elements of the kernel in terms of classes in $H^0(S,\mathcal{A}^*/\mathcal{A}^{*G})$, using the long exact sequence of sheaf cohomology. Each class $[f]$ with f defined on an open cover $\mathcal{U}$ ($f_U\in\mathcal{A}(U)$ and $f_U|_{U\cap V} \equiv f_V|_{V\cap U} \bmod \mathcal{A}(U\cap V)^{*G}$ for $U,V\in\mathcal{U}$) defines a cohomology class $[\gamma]$ in $H^1(G,\mathcal{A}(S)^*)$, where γ is the cocycle with

$$\gamma(g)|_U = f_U(gf_U)^{-1} \quad (g\in G, U\in\mathcal{U}).$$

By definition γ is localy trivial and the process is clearly reversible.

A second proof can be obtained by noting that the functors 'global sections' and 'G-fixed points' have the same composition in either order when applied to the category of sheaves of $\mathbb{Z}G$-modules on S. Each composition gives here a Grothendieck spectral sequence for the composite functor, whose derived groups we denote by $H^n(G,\mathcal{M})$ at a sheaf $\mathcal{M}$ of $\mathbb{Z}G$-modules on S [11]. In particular we have two exact sequences

$$0 \to H^1(G,\mathcal{M}(S)) \to H^1(G,\mathcal{M}) \to H^1(S,\mathcal{M})^G$$

and

$$0 \to H^1(S, \mathcal{M}^G) \to H^1(G, \mathcal{M}) \to H^0(S, \mathcal{H}^1(G, \mathcal{M})),$$

where $\mathcal{H}^1(G, \mathcal{M})$ is the sheafification of the presheaf which assigns an open set U of S to $H^1(G, \mathcal{M}(U))$. Using appropriate standard arguments with injectives, one shows that the composites $H^1(S, \mathcal{M}^G) \to H^1(S, \mathcal{M})^G$ and $H^1(G, \mathcal{M}(S)) \to H^0(S, \mathcal{H}^1(G, \mathcal{M}))$ are indeed the obvious maps. The identification of their kernels gives the desired result, when interpreted for $\mathcal{M} = \mathcal{A}^*$.

The spectral sequences above appear to generalize the theory of Chase, Harrison, and Rosenberg [3, Corollary 5.5], though we have not pursued this.

In case G acts trivially on the units of $\mathcal{A}(S)$, then $H^1(H, \mathcal{A}(S)^*)$ is just $\mathrm{Hom}_{\mathrm{group}}(H, \mathcal{A}(S)^*)$ for each subgroup H of G. Consequently, if $\{H_\alpha\}$ is a collection of subgroups generating G with S-Pic $\mathcal{A}^{H_\alpha} \to S$-Pic $\mathcal{A}$ injective for each α, then S-Pic $\mathcal{A}^G \to S$-Pic $\mathcal{A}$ is injective. This principle was applied very effectively by Kang [8] to rings of invariants in polynomial rings. Here is an analgous reduction which does not require trivial action, and instead relies on properties of the Hecke category.

4.7.3. Proposition. *Let $S, G, \mathcal{A}$ be as in Proposition 4.7.1 and suppose $\{H_\alpha\}$ is a family of subgroups of finite index in G with $\mathrm{GCD}_\alpha \{[G:H_\alpha]\} = 1$. Then the natural map*

$$S\text{-Pic}\, \mathcal{A}^G \to \prod_\alpha S\text{-Pic}\, \mathcal{A}^{H_\alpha}$$

is injective.

This follows from Proposition 2.1 and the fact that the map $\bigoplus_\alpha \mathbb{Z}G/H_\alpha \to \mathbb{Z} = \mathbb{Z}G/G$ is split. As in Corollary 4.7.2, one can immediately specialize to $\mathrm{Pic}\, A^G$.

Actually, it is somewhat sharper to formulate this proposition a prime at a time: Let p be fixed prime and let $M_{(p)}$ denote the localization at the prime ideal (p) of $\mathbb{Z}$ for a $\mathbb{Z}$-module M.

4.7.4. *Let $S, G, \mathcal{A}$ be as in Proposition 4.7.1 and suppose H is a subgroup of finite index in G with $[G:H]$ not divisible by p. Then the map*

$$(S\text{-Pic}\, \mathcal{A}^G)_{(p)} \to (S\text{-Pic}\, \mathcal{A}^H)_{(p)}$$

is injective.

This follows because the composite $\mathbb{Z} \to \mathbb{Z}G/H \to \mathbb{Z}$ is multiplication by $[G:H]$, hence the map $\mathbb{Z}_{(p)}G/H \to \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}G/G$ is split.

One could also view 4.7.4 as a consequence of the norm theory of 4.3. This theory can also be used, exactly as in the analogous situation for the cohomology of finite groups [20, XII, 10.1] to describe the image of the map in 4.7.4 as the set of 'stable' classes in $(S\text{-Pic}\, \mathcal{A}^H)_{(p)}$: those whose conjugates under any given $g \in G$ give the same restriction to $(S\text{-Pic}\, \mathcal{A}^{H \cap gHg^{-1}})_{(p)}$ as the class itself.

4.8. *Variations on* Pic A. It seems likely that the methods of this paper extend without difficulty to Cartier class groups and possibly other kinds of 'fractional ideals'. We have not, however, considered Azumaya algebras or the Brauer group of a scheme, and leave this project for an interested reader familiar with [6]. It might be possible in some cases to do something with K_0, using roughly the method discussed in 4.1. for one-sided modules. This should become clearer after the issues raised in 4.1 have been more thoroughly investigated.

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